

Geometry II Final Exam Solutions 2006-2007

Q1:

a)

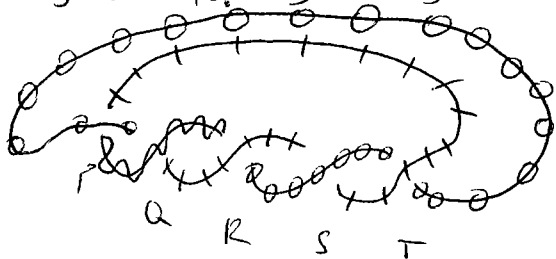


Similar to exercise

3 marks

P has all one colour $\Rightarrow$ Q all one colour $\Rightarrow$ same for R

$\Rightarrow$ same for S $\Rightarrow$ same for T. Only one colour used in diagram $\Rightarrow$ not a valid tricolouring.



4 marks

P coloured as shown $\Rightarrow$ Q coloured as shown

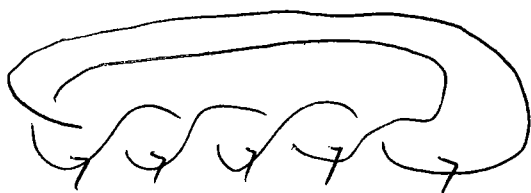
$\Rightarrow$ R coloured as shown $\Rightarrow$ 2 colours at S and T.

$\Rightarrow$ not a valid tricolouring.

Therefore, the knot diagram is not tricolourable. 1 mark

b) Put in orientation:

Similar to exercise



①

There are 5 crossings. Each has $\text{sgn} = -1$. ②

The writhe is $-1 - 1 - 1 - 1 - 1 = -5$. Done in class ②

c) After Reid move II



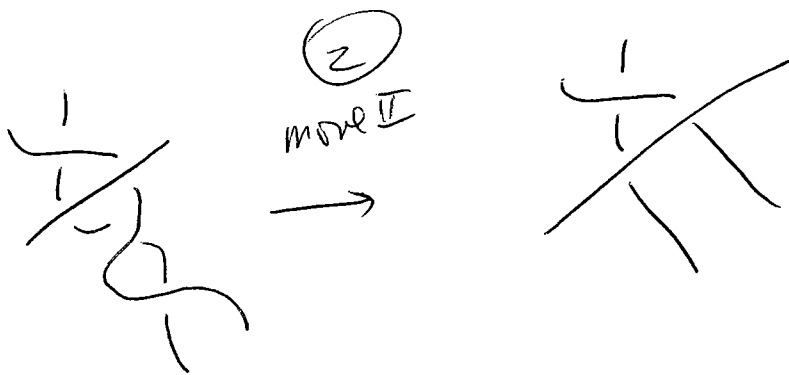
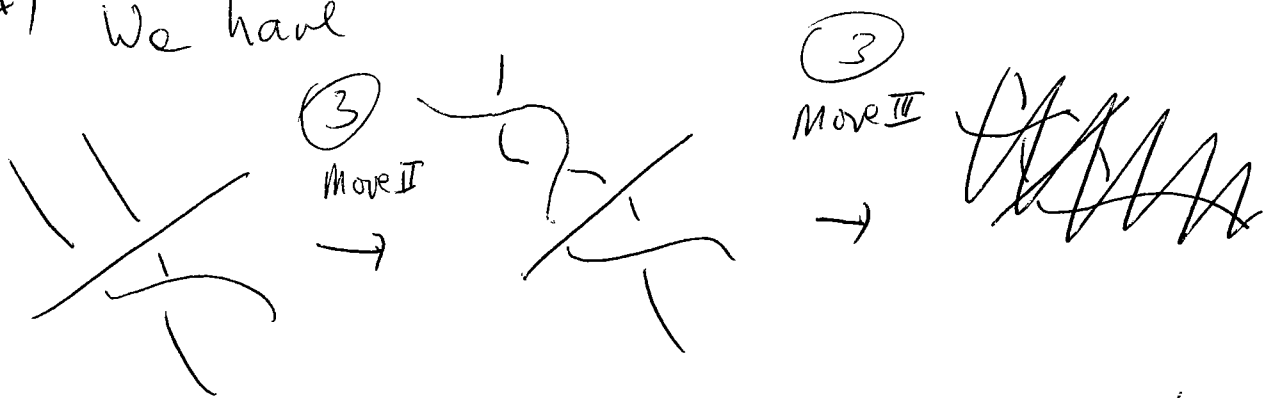
becomes



where $\square$ denotes rest of knot. $\text{sgn} P = -1$ $\text{sgn} Q = +1$ ②

so $\text{Writhe}(\square \text{ with } P \text{ and } Q) = \text{Writhe}(\square) + 1 - 1 = \text{Writhe}(\square)$ ③

d) We have



done
in
class

(we can also do the moves backwards)

Q2 All done in class

a) $J(x) = B(x)(-x)^{3W}$

$J_R(x) = J(x^{-1})$

Let $t = x^4$

② done
in
class
①

$$\begin{aligned}
 b) \quad B\left(\bigcirc_P\right) &= x B(\bigcirc) + x^{-1} B(\infty) \\
 &= x \left[x B(00) + x^{-1} B(\infty) \right] \\
 &\quad + x^{-1} \left[x B(0) + x^{-1} B(00) \right] \\
 &= x \left[-x(x^2 + x^{-2}) + x^{-1} \right] \\
 &\quad + x^{-1} \left[x - (x^2 + x^{-2})x^{-1} \right] \\
 &= x(-x^3) + x^{-1}(-x^{-3}) = -x^{-4} - x^{-4}. \quad (6)
 \end{aligned}$$

$$\begin{aligned}
 c) \quad B\left(\bigcirc_P\right) &= x^{-1} B(\infty) + x B(\bigcirc) \\
 &= x^{-1}(-x^5 - x^{-3} - x^{-7}) + x(-x^3) B(\bigcirc) \\
 &= x^{-1}(-x^5 - x^{-3} - x^{-7}) - x^4(-x^4 - x^{-4}) \\
 &= 1 - x^4 - x^{-4} + x^8 + x^{-8}. \quad (7)
 \end{aligned}$$

d) Put in orientation: 

$$\text{sgn } P = -1, \text{sgn } Q = -1$$

$$\text{sgn } R = +1, \text{sgn } S = +1 \quad \text{Writhe} = 1 + 1 - 1 - 1 = 0$$

$$B(x)(-x)^{3W} = B(x). \quad J(t) = 1 - t - t^{-1} + t^2 + t^{-2}.$$

(6)

$$\begin{aligned}
 e) \quad J_R(t) &= J(t^{-1}) = 1 - t^{-1}(t^{-1})^{-1} + (t^{-1})^2 + (t^{-1})^{-2} \\
 &= 1 - t^{-1} - t + t^{-2} + t^2 \\
 &= J(t).
 \end{aligned}$$

Q3: a) $\gamma(t)$ is regular if $\dot{\gamma}(t) \neq 0$ for all $t \in (a, \beta)$. ^{done in class} (2)

$$\dot{\gamma}(t) = (-2t \sin(t^2), 2t \cos(t^2), 2t)$$

$$\|\dot{\gamma}(t)\| = \|2t(-\sin(t^2), \cos(t^2), 1)\|$$

$$= 2|t|\sqrt{2}.$$

similar to exercises

$\|\dot{\gamma}(t)\| = 0$ if $t = 0$ so γ is not regular (3)

b) $\tilde{\gamma}(\tilde{t})$ is a reparametrization of $\gamma(t)$ if $\exists$ ^{done in class} a function $\phi: (\tilde{a}, \tilde{\beta}) \rightarrow (a, \beta)$ with smooth inverse such that $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$. (3)

By the chain rule, we have

$$\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt} \frac{d\phi}{d\tilde{t}} \quad (1)$$

$$\text{so } \left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = \left\| \frac{d\gamma}{dt} \right\| \left| \frac{d\phi}{d\tilde{t}} \right|. \quad (1)$$

Moreover $\phi^{-1}(\phi(\tilde{t})) = \tilde{t} \Rightarrow \frac{d\phi^{-1}}{d\phi} \frac{d\phi}{d\tilde{t}} = 1 \Rightarrow \left| \frac{d\phi}{d\tilde{t}} \right| > 0 \quad \forall \tilde{t} \in (\tilde{a}, \tilde{\beta})$. (2)

~~The~~ Therefore $\left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| > 0$ iff $\left\| \frac{d\gamma}{dt} \right\| > 0$ (1)

so $\tilde{\gamma}$ is normal iff γ is normal. (1)

$$c) \quad \tau = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \quad \beta = \frac{\dot{\gamma} \wedge \ddot{\gamma}}{\|\dot{\gamma} \wedge \ddot{\gamma}\|}, \quad v = \beta \times \tau.$$

done in class

$$\dot{\tau} = \kappa v$$

(5)

$$\dot{v} = \cancel{\kappa \tau} - \kappa \tau + \tau \beta$$

$$\dot{\beta} = -\tau v$$

$$d) \quad \dot{\gamma}(t) = \left(\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\|\dot{\gamma}(t)\|^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2} = 1$$

$\Rightarrow \gamma$ unit-speed

$$\ddot{\gamma}(t) = \left(\frac{1}{4}(1+t)^{-\frac{1}{2}}, \frac{1}{4}(1-t)^{-\frac{1}{2}}, 0 \right)$$

$$\begin{aligned} \kappa = \|\ddot{\gamma}(t)\| &= \sqrt{\frac{1}{16(1+t)} + \frac{1}{16(1-t)}} \\ &= \frac{1}{4} \sqrt{\frac{1}{1+t} + \frac{1}{1-t}} = \frac{1}{4} \sqrt{\frac{2}{1-t^2}}. \end{aligned}$$

$$\tau = \dot{\gamma}(t) \text{ above}$$

$$\begin{aligned} v &= \frac{\ddot{\gamma}(t)}{\|\ddot{\gamma}(t)\|} = \frac{4}{\sqrt{2}} \sqrt{1-t^2} \left(\frac{1}{4}(1+t)^{-\frac{1}{2}}, \frac{1}{4}(1-t)^{-\frac{1}{2}}, 0 \right) \\ &= \frac{1}{\sqrt{2}} \left(\sqrt{1-t}, \sqrt{1+t}, 0 \right) \end{aligned}$$

$$\beta = \tau \times v = \frac{1}{2\sqrt{2}} \det \begin{pmatrix} i & j & k \\ \frac{1}{\sqrt{1+t}} & -\frac{1}{\sqrt{1-t}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{1-t}} & \frac{1}{\sqrt{1+t}} & 0 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \left((1+t) + (1-t), \frac{1}{\sqrt{2}} \sqrt{1-t}, \frac{1}{\sqrt{2}} \sqrt{1+t} \right)$$

$$= \frac{1}{2\sqrt{2}} \left(-\sqrt{2}\sqrt{1+t}, \sqrt{2}\sqrt{1-t}, (1+t)+(1-t) \right)$$

$$= \frac{1}{2} \left(-\sqrt{1+t}, \sqrt{1-t}, \sqrt{2} \right). \quad (6)$$

Q4

$$a) \quad K_S \hat{k} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\|\dot{\vec{r}}\|^3}$$

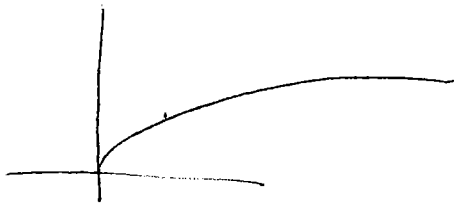
(1)

done in class

$$|K_S| = \|K_S \hat{k}\| = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = K. \quad (3)$$

Like exercises

b) i)



$$(ii) \quad \dot{\vec{r}} = \left(1, \frac{1}{2t}, 0 \right), \quad \tau = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|} = \frac{1}{\sqrt{1 + \frac{1}{4t}}} \left(1, \frac{1}{2t}, 0 \right)$$

$$\tau(1/4) = \frac{1}{\sqrt{2}} (1, 1, 0)$$

$$\lim_{t \rightarrow \infty} \tau(t) = (1, 0, 0)$$

$$(iii) \quad \ddot{\vec{r}} = \left(0, -\frac{1}{4t^{3/2}}, 0 \right) \quad \ddot{\vec{r}} \times \dot{\vec{r}} = -\frac{1}{4t^{3/2}} \hat{k}$$

$$\frac{\ddot{\vec{r}} \times \dot{\vec{r}}}{\|\dot{\vec{r}}\|^3} = -\frac{\frac{1}{4t^{3/2}} \hat{k}}{\left(1 + \frac{1}{4t}\right)^{3/2}} \Rightarrow K_S = -\frac{\frac{1}{4t^{3/2}}}{\left(1 + \frac{1}{4t}\right)^{3/2}}. \quad \text{Angle change} = -\frac{\pi}{4}$$

$$\begin{aligned} \int_{\frac{1}{4}}^{\infty} K_S \|\dot{\vec{r}}\| dt &= \int_{\frac{1}{4}}^{\infty} -\frac{1}{4t^{3/2} \left(1 + \frac{1}{4t}\right)^{3/2}} \left(1 + \frac{1}{4t}\right)^{1/2} dt = \int_{\frac{1}{4}}^{\infty} -\frac{1}{4t^{3/2} \left(1 + \frac{1}{4t}\right)} dt = \int_{\frac{1}{4}}^{\infty} -\frac{1}{\sqrt{t}(4t+1)} dt \\ &= -\frac{\pi}{4} \Rightarrow \int_{\frac{1}{4}}^{\infty} \frac{1}{\sqrt{t}(4t+1)} dt = \frac{\pi}{4}. \end{aligned}$$

$$c) T = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \ddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|^2}$$

done in class

if $\mathbf{r}$ is planar, then $\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \ddot{\mathbf{r}}$ are contained in the same plane $\Rightarrow \ddot{\mathbf{r}} = a(t)\dot{\mathbf{r}} + b(t)\ddot{\mathbf{r}}$ for some functions $a(t), b(t) \Rightarrow$

$$(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \ddot{\mathbf{r}} = \cancel{a(t)\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + b(t)\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}$$

$$= (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot (a(t)\dot{\mathbf{r}} + b(t)\ddot{\mathbf{r}})$$

$$= a(t)(\dot{\mathbf{r}} \times \dot{\mathbf{r}}) \cdot \ddot{\mathbf{r}} + b(t)(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \ddot{\mathbf{r}}$$

$$= a(t) \cdot 0 + b(t) \cdot 0 = 0.$$

$$\text{So } T = 0.$$

(6)

QS

$$a) \sigma_u = (1, 0, 2u) \quad \sigma_v = (0, 1, 2v)$$

$$\sigma_u \times \sigma_v = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{pmatrix} = (-2u, -2v, 1)$$

$$N_\sigma = \frac{(-2u, -2v, 1)}{\sqrt{4u^2 + 4v^2 + 1}}$$



Similar to exercises

$$b) \hat{\sigma}_r = (\cos\theta, \sin\theta, 2r) \quad \hat{\sigma}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$N_{\hat{\sigma}} = \hat{\sigma}_r \times \hat{\sigma}_\theta = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$N_{\hat{\sigma}} = \frac{(-2r^2\cos\theta, -2r^2\sin\theta, r)}{\sqrt{4r^4 + r^2}} = \frac{(-2r\cos\theta, -2r\sin\theta, 1)}{\sqrt{4r^2 + 1}}$$

if $u = r\cos\theta, v = r\sin\theta$, then $N_\sigma = N_{\hat{\sigma}}$ because $4u^2 + 4v^2 = 4r^2$.

$$c) \quad E = \tilde{\sigma}_r \cdot \tilde{\sigma}_r = 1 + 4r^2$$

$$F = \tilde{\sigma}_r \cdot \tilde{\sigma}_\theta = 0$$

$$G = \tilde{\sigma}_\theta \cdot \tilde{\sigma}_\theta = r^2$$

$$\tilde{\sigma}_{rr} = (0, 0, 2) \quad \tilde{\sigma}_{\theta\theta} = (-r\cos\theta, -r\sin\theta, 0)$$

$$\tilde{\sigma}_{r\theta} = (-\sin\theta, \cos\theta, 0)$$

$$L = \tilde{\sigma}_{rr} \cdot N_{\tilde{\sigma}} = \frac{2}{\sqrt{4r^2+1}}$$

$$M = \tilde{\sigma}_{r\theta} \cdot N_{\tilde{\sigma}} = \cancel{\text{scribbles}} \quad 0$$

$$N = \tilde{\sigma}_{\theta\theta} \cdot N_{\tilde{\sigma}} = \frac{2r^2}{\sqrt{4r^2+1}}$$

(this will be hard for them)

$$d) \quad z \leq 2 \text{ corresponds to } r \leq \sqrt{2}$$

Surface area of part with $r \leq \sqrt{2}$ equals

$$\int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{EG - F^2} \, d\theta \, dr = \int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{(1+4r^2)r^2} \, d\theta \, dr$$

$$= 2\pi \int_0^{\sqrt{2}} r \sqrt{1+4r^2} \, dr = \frac{2\pi}{12} (1+4r^2)^{3/2} \Big|_{r=0}^{r=\sqrt{2}}$$

$$= \frac{\pi}{6} \left[(1+4 \cdot 2)^{3/2} - (1+0)^{3/2} \right] = \frac{\pi}{6} \left[9^{3/2} - 1^{3/2} \right]$$

$$= \frac{\pi}{6} [27 - 1] = \frac{26\pi}{6} = \frac{13\pi}{3}$$

$$b) a) \quad \sigma_u = (\cos v, \sin v, 0)$$

$$\sigma_v = (-u \sin v, u \cos v, 1)$$

$$E = \sigma_u \cdot \sigma_u = 1$$

$$F = \sigma_u \cdot \sigma_v = 0$$

$$G = \sigma_v \cdot \sigma_v = u^2 + 1$$

$$b) \quad 1 = \|\ddot{\gamma}\|^2 = E \dot{u}^2 + 2F \dot{u} \dot{v} + G \dot{v}^2 \\ = \dot{u}^2 + (1+u^2) \dot{v}^2$$

$$c) \quad \frac{d}{dt} (0 + (1+u^2) \dot{v}) = \frac{1}{2} (0 + 0 + 0) \\ = 0$$

$$\Rightarrow (1+u^2) \dot{v} = a \text{ for constant } a$$

$$\Rightarrow \dot{v} = \frac{a}{1+u^2}$$

$$d) \quad \left(\frac{dv}{du} \right)^2 = \left(\frac{\dot{v}}{\dot{u}} \right)^2 = \frac{\dot{v}^2}{\dot{u}^2}$$

$$= \frac{\dot{v}^2}{1 - (1+u^2) \dot{v}^2} = \frac{\frac{a^2}{(1+u^2)^2}}{1 - (1+u^2) \frac{a^2}{(1+u^2)^2}}$$

$$= \frac{a^2}{(1+u^2)^2} \cdot \frac{1+u^2}{1+u^2-a^2} = \frac{a^2}{(1-a^2+u^4)(1+u^2)}$$

e) if $a=0$ $\frac{dv}{du} = 0 \Rightarrow v$ constant

if $a=1$ $\left(\frac{dv}{du}\right)^2 = \frac{1}{u^2(1+u^2)} \quad (*)$

Let $V = V_0 + \sinh^{-1}\left(\frac{1}{u}\right)$

then $\frac{dv}{du} = 0 + \frac{1}{\sqrt{1 + \frac{1}{u^2}}} \cdot \left(-\frac{1}{u^2}\right)$

$$\left(\frac{dv}{du}\right)^2 = \frac{1}{1 + \frac{1}{u^2}} \cdot \frac{1}{u^4}$$

$$= \frac{1}{u^2(1+u^2)}, \text{ agreeing with } (*)$$